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Global structures for the moduli of (punctured) super Riemann surfaces ★

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Abstract

A fine moduli superspace for algebraic super Riemann surfaces with a level-n structure is constructed as a quotient of the split superscheme of local spin-gravitivo fields by an étale equivalence relation. This object is not a superscheme, but still has an interesting structure: it is an algebraic superspace, that is, an analytic superspace with sufficiently many meromorphic functions. The moduli of super Riemann surfaces with punctures (fixed points in the supersurface) is also constructed as an algebraic superspace. Moreover, when one only considers ordinary punctures (fixed points in the underlying ordinary curve), it turns out that the moduli is a true superscheme. We prove furthermore that this moduli superscheme is split.

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1. Introduction

The computation of the quantum scattering vacuum amplitudes in Polyakov's bosonic string theory [21] requires an integration over a compactification of the moduli space of Riemann surfaces that represents the string world sheet. Furthermore, the matrix elements of vertex operators, that appear as points of the world sheet, provides the N-point Green functions of the theory, thus requiring that the right space of integration to compute the

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correlation functions of the theory is a compactification of the moduli space of punctured Riemann surfaces (Riemann surfaces with a fixed family of ordered points).

The compactification of the moduli space has the effect that Polyakov's measure has poles at the boundary. One of the reasons behind the introduction of supersymmetry in string theory was to be able to define a new measure with a fermionic component that compensated the poles. Friedan suggested the supermoduli or moduli space of super Riemann surfaces, briefly called SUSY curves, as the natural framework for the formulation of the amplitude integral in superstring theory [13]. This supermoduli has been widely considered in the physical literature on supersymmetry, but its global geometric structure still lacks of a systematic study. There are analytic constructions of local nature that give the supermoduli space as a super-orbifold (see [4,18] and [2,3,14] for similar descriptions for punctured SUSY curves). The letter [5] contains the necessary ideas to obtain globally a moduli superspace as an algebraic superstack constructed from Deligne–Mumford's moduli algebraic stack of stable smooth curves [6]. A coordinate description of a similar kind of moduli superspace, without attempting to formalize its structure, is contained in [11].

In this paper we give a new construction of a supermoduli space for SUSY curves as an *algebraic superspace*, with techniques and methods of Algebraic Geometry; this allows for global constructions in a natural way. Algebraic superspaces are defined here for the first time, and they are the graded objects corresponding to Artin's algebraic spaces [1,16].

As Deligne suggested, the category of superschemes, that is, schemes with a graded structure in the sense of Berezin–Leĭtes–Kostant, is not wide enough to contain the supermoduli spaces of SUSY curves. The larger category of algebraic superspaces seems to be the natural arena for that problem: Artin's algebraic superspaces are quotients of étale equivalence relations of superschemes, then very particular instances of algebraic superstacks, which means that the category of algebraic superstacks is too big for our purposes. Moreover, algebraic superspaces still enjoy a rich geometric structure: In the non-graded case, algebraic spaces have an underlying analytic space with sufficiently many meromorphic functions, more precisely, they are Moisezon spaces, that is, analytic spaces whose field of meromorphic functions has trascendence degree equal to the dimension. Moreover, Moisezon proved that every Moisezon space is the underlying analytic space to an Artin's algebraic space [19]. One might then think of algebraic superspaces as analytic superspaces with a sufficiently large field of meromorphic superfunctions.

We always consider proper smooth curves of genus $g \geq 2$ with a level-$n \geq 3$ structure. Our first main result is:

(1) There is a *fine moduli space* in the category of algebraic superspaces for the sheaf functor $\mathcal{S}$ of SUSY curves; in other words, $\mathcal{S}$ is isomorphic to the functor of the points of an algebraic superscheme. Moreover, the moduli superspace has dimension $(3g - 3, 2g - 2)$.

The next step is to consider NS (Neveu–Schwarz) punctures the SUSY curves in the [3,14]. sense of For a relative SUSY curve, that is, a family of SUSY curves depending on bosonic and fermionic parameters, a NS puncture is merely a section of the family. It can be locally described in relative conformal coordinates as 'fixing a graded point', $(z, \theta) \mapsto (z_0, \theta_0)$. This imposes one more even and one more condition to automorphisms of a neighborhood around the marked graded point that are allowed. The Lie algebra of conformal supervector fields is changed, and within the operators formalism, the change is interpreted as the insertion of a vertex operator at the point. It has also the effect of increasing by one both the even and the odd dimension of the supermoduli. Our second result is:

(2) Let us consider the sheaf functor $\mathcal{S}^N$ of N-punctured SUSY curves, that is, SUSY curves with N graded fixed points (sections). There is also a fine moduli space for this functor in the category of algebraic superspaces. The moduli superspace for N-punctured SUSY curves has dimension $(3g-3+N, 2g-2+N)$.

We can achieve a supermoduli space which is a true superscheme if we introduce the notion of $\widetilde{N}$-punctured SUSY curves. For a family of SUSY curves depending on bosonic and fermionic parameters, a $\widetilde{1}$-puncture consists of fixing a section of the underlying ordinary family of curves. It can be locally described as 'fixing an ordinary point', $z \mapsto z_0$. The supermoduli for $\widetilde{1}$-punctured SUSY curves depends then on one more even parameter, though the odd dimension remains unchanged.

$\widetilde{N}$-punctures are introduced merely for technical reasons, as no previous physical correspondent is known to us. One might think that they correspond to insertion points of bosonic vertex operators. Our third result is:

(3) The sheaf functor $\mathcal{S}^{\widetilde{N}}$ of $\widetilde{N}$-punctured SUSY curves is representable by a true superscheme of dimension $(3g-3+N, 2g-2)$. Moreover, this fine moduli superscheme is *split*, that is, its structure sheaf is an exterior algebra.

The last result gives then a partial answer to the open question of the splitness of the moduli space for SUSY curves [10,11,18].

There are two problems related with the integration over these moduli algebraic superspaces. The first one is finding the right compactification. The natural candidate is the super correspondent for Deligne–Mumford compactification of the moduli of smooth curves by means of stable curves. We do not deal with this question in this paper, as we always consider proper smooth curves. A second problem is the definition of a suitable Polyakov's supermeasure on the moduli superspace. At this point the introduction of the category of Artin's algebraic superspaces proves its usefulness. In former approaches, moduli superspaces are super-orbifolds, and it is not easy to define volume superforms on them. The rich geometry of Artin's algebraic superspaces enables us to introduce determinant bundles following the bosonic model. The idea is to consider the universal SUSY curve $\pi : \mathcal{X} \to \mathcal{M}$ over the moduli algebraic superspace $\mathcal{M}$ and define the berezinian determinant bundles as

$$\Lambda_i = \mathrm{Ber}(\mathbb{R}\pi_* \omega^{\otimes i}),$$

where ω stands for the relative dualizing sheaf of π. One can conjecture that a Mumford formula $\Lambda_i = (\Lambda_1)^{\otimes n(i)}$ still holds. Moreover there is an isomorphism $\mathrm{Ber}(\Omega^1_{\mathcal{M}}) \xrightarrow{\sim} \Lambda_2$, given by the computation of the infinitesimal deformations, so that one could define integration supermeasures on $\mathcal{M}$ as sections of a certain sheaf $\Lambda_i \otimes \bar{\Lambda}_i$. Then, the integration problem on the moduli superspace would be reduced to a berezinian integration on the underlying analytic superspace.

The introduction of level-n structures is a rigidification that implies the existence of fine moduli algebraic superspaces. Level-n structures are the algebraic analogous to homological markings on the curve (roughly speaking, fixing a basis 'modulo n' of the homology) and have the effect of restricting the automorphisms that are allowed. When one considers the moduli of curves as a subspace of the moduli of abelian varieties, level-n structures come out when one considers modular forms and theta functions with characteristics. In the physical literature, theta functions with characteristics are related to the construction of solutions of soliton-type to the motion equations. One may conjecture that, if we do not fix a level structure, there is a coarse moduli algebraic superspace whose (closed) points correspond to punctured SUSY curves. Our fine moduli algebraic superspace will then be a finite covering of this coarse moduli algebraic superspace.

Another important question still pending, is the algebraic interpretation of the R (Ramond) -punctures or spin nodes [3,14], that seems to be related with the compactification by stable supercurves of the moduli algebraic superspaces of punctured SUSY curves.

The work is in progress in these directions.

The paper is organized as follows: In Section 2 we summarize the basic definitions about SUSY curves (classical or punctured) in the category of superschemes. SUSY curves are then families of supercurves whit a conformal structure, and we consider different types of punctures on them: N-punctures, that is, points (sections) in the family of supercurves, corresponding to bosonic–fermionic superstring states, and $\widetilde{N}$-punctures, that is, points (sections) in the underlying family of ordinary curves corresponding to bosonic string states. We also give the definitions of étale morphism and étale covering for superschemes that enable us to consider an étale topology of superschemes. This will be needed in the subsequent sections, mainly in connection with the representability problems involved.

In Section 3 we discuss these objects for the category of ordinary schemes. We define the functor of spin curves, that is, (relative) curves $X \to Y$ with a (relative) spin structure $\kappa_{X/Y}^{1/2}$, i.e., a square root of the (relative) canonical sheaf. For curves of genus $g \geq 2$ with a level-n structure ($n \geq 3$), we prove the existence of a fine moduli scheme M_s of dimension $3g - 3$ with a universal curve $X_s \to M_s$. From this fact, we deduce that the N-fold fiber product of X_s over M_s represents the functor of $\widetilde{N}$-punctured spin curves.

In Section 4 we define the functor of SUSY curves and we prove that the restriction of this functor to the category of ordinary schemes is simply the aforementioned functor of spin curves. Analogous statements for punctured SUSY curves are also considered.

Section 5 is devoted to the study of the properties that a superscheme $\mathcal{M}$ representing the functor of SUSY curves should have if it would exist. We start with the proof that the underlying ordinary scheme to such a $\mathcal{M}$ is the moduli scheme M_s, and then we describe $\mathcal{M}$ locally from its infinitesimal deformations. This implies that $\mathcal{M}$ must be locally isomorphic with the superscheme of local *spin–gravitivo fields* $(M_s, \bigwedge \kappa_s^{3/2})$, where $(X_s, \kappa_s^{1/2})$ is a (local) universal spin curve over M_s, and $\kappa_s^{3/2} = \kappa_s \otimes \kappa_s^{1/2}$ is the locally free sheaf of rank $2g-2$, whose sections are the gravitino fields. We finally quote two fundamental theorems from [18] that are on the base of the subsequent constructions.

Section 6 contains the first two main results: there is an étale equivalence relation in the functor of points of the local spin–gravitino structures, whose quotient is precisely the sheaf functor of SUSY curves $\mathcal{S}$. This does not imply that $\mathcal{S}$ is representable in the category of superschemes, but leaves us in an analogous situation to the one that leads to the definition of Artin's algebraic spaces. Following Artin's idea, we give the basic definitions of the new category of *algebraic superspaces* and restate the previous results as a theorem of representability for $\mathcal{S}$ by an algebraic superspace of dimension $(3g-3, 2g-2)$. We also prove that there exists a fine moduli space for the functor $\mathcal{S}^N$ of N-punctured SUSY curves in the category of algebraic superspaces and that it has dimension $(3g-3+N, 2g-2+N)$.

In Section 7 we consider the moduli scheme X_s^N of $\widetilde{N}$-punctured spin curves. There is a universal spin structure $(\bar{X}_s^N/X_s^N, \bar{\Upsilon}_s)$ and we construct in a natural way an invertible sheaf $\bar{L}$ representing $\bar{\Upsilon}_s$ from the theta divisor of the Jacobian. This enables us to define a split superscheme $\mathcal{X}_s^N = (X_s^N, \bigwedge \pi_*(\bar{L}\otimes\kappa))$ which has dimension $(3g-3+N, 2g-2)$. We finally prove that this superscheme is a fine moduli scheme for the functor of $\widetilde{N}$-punctured SUSY curves.

Some of the results in this paper generalize theorems in [7].

2. SUSY curves over algebraic superschemes

We consider only superschemes $\mathcal{X} = (X, \mathcal{A})$ in the sense of Berezin–Leĭtes–Kostant, whose underlying scheme X is of finite type over the field of the complex numbers.

By a curve X/Y we always mean a proper smooth morphism $X \to Y$ of relative dimension 1.

We shall denote by $\mathcal{X}/\mathcal{Y}$ a family $\pi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ of proper smooth supercurves of (relative) dimension $(1, 1)$ and fixed genus g, in the sense that the underlying curve X/Y has genus g (see, for instance, [9, Definition 3]).

Definition 2.1. A conformal structure $\mathcal{D}$ over $\mathcal{X}/\mathcal{Y}$ is a locally free submodule $\mathcal{D}$ of rank $(0, 1)$ of the relative tangent sheaf $\operatorname{Der}\mathcal{A}/\mathcal{B}$ such that the map

$$\mathcal{D}\otimes_{\mathcal{A}}\mathcal{D} \xrightarrow{[,] \bmod \mathcal{D}} (\operatorname{Der}\mathcal{A}/\mathcal{B})/\mathcal{D}$$

is an isomorphism of $\mathcal{A}$-modules.

The couple $(\mathcal{X}/\mathcal{Y}, \mathcal{D})$ is called a super Riemann surface, a super symmetric curve, or briefly, a SUSY curve [9, Definition 8;18].

When $\mathcal{Y} = Y$ is an ordinary scheme, then $\mathcal{X}$ is trivial, in the sense that $\mathcal{X} = (X, \bigwedge_{\mathcal{O}} L)$ for some invertible sheaf L over $(X, \mathcal{O})$. Then the existence of a conformal structure $\mathcal{D}$ on $\mathcal{X}/Y$ is equivalent to the existence of an isomorphism of $\mathcal{O}$-modules $L\otimes_{\mathcal{O}} L \simeq \kappa_{X/Y}$, where $\kappa_{X/Y}$ is the (relative) canonical sheaf, that is, L is a spin structure $L = \kappa_{X/Y}^{1/2}$ on the underlying scheme X. $\mathcal{D}$ and L are related by the expression $\mathcal{D}\otimes_{\mathcal{A}}\mathcal{O} \simeq L^{-1}$.

A morphism of supercurves over $\mathcal{Y}$, $\phi\colon \mathcal{X}'/\mathcal{Y} \to \mathcal{X}/\mathcal{Y}$, induces a morphism of $\mathcal{B}$-modules $\phi_*\colon \operatorname{Der}\mathcal{A}'/\mathcal{B} \to \phi^*\operatorname{Der}\mathcal{A}/\mathcal{B} = \operatorname{Der}\mathcal{A}/\mathcal{B}\otimes_{\mathcal{A}}\mathcal{A}'$, and then, a conformal structure $\mathcal{D}'$ on $\mathcal{X}'/\mathcal{Y}'$ defines a subsheaf $\phi_*\mathcal{D}'$ of $\operatorname{Der}\mathcal{A}/\mathcal{B}\otimes_{\mathcal{A}}\mathcal{A}'$.

Definition 2.2. Let $(\mathcal{X}/\mathcal{Y}, \mathcal{D})$ and $(\mathcal{X}'/\mathcal{Y}, \mathcal{D}')$ be SUSY curves over a superscheme $\mathcal{Y}$. A morphism of SUSY curves over $\mathcal{Y}$ is a morphism of supercurves over $\mathcal{Y}$, $\phi\colon \mathcal{X}'/\mathcal{Y} \to \mathcal{X}/\mathcal{Y}$, such that $\phi_*\mathcal{D}' \subseteq \mathcal{D} \otimes_{\mathcal{A}} \mathcal{A}'$.

We can then consider the category of super Riemann surfaces or SUSY curves of genus g over a superscheme $\mathcal{Y}$.

The automorphisms of a SUSY curve $(\mathcal{X}/\mathcal{Y}, \mathcal{D})$ are the automorphisms τ of $\mathcal{X}/\mathcal{Y}$ as a supercurve over $\mathcal{Y}$ such that $\mathcal{D}$ and $\tau_*\mathcal{D}$ define the same subsheaf of $\operatorname{Der} \mathcal{A}/\mathcal{B}$. In particular, for a SUSY curve over an ordinary scheme, that is, a supercurve of type $(X, \bigwedge_{\mathcal{O}} L)/Y$ together with an isomorphism of $\mathcal{O}$-modules $L \otimes_{\mathcal{O}} L \xrightarrow{\sim} \kappa_{X/Y}$, an automorphism is merely a pair $\tau = (\tau_0, \tau_1)$, where τ_0 is an automorphism of the underlying family of curves X/Y and τ_1 is an automorphism of L as a sheaf of complex vector spaces, such that the isomorphism $L \otimes_{\mathcal{O}} L \xrightarrow{\sim} \kappa_{X/Y}$ is preserved (in particular, if $\tau_0 = \mathrm{Id}$ then $\tau_1 = \pm\,\mathrm{Id}$).

Definition 2.3. A N-puncture on a supercurve $\pi\colon \mathcal{X} \to \mathcal{Y}$ is an ordered family $(s_1, \ldots, s_N)$ of sections $s_i\colon \mathcal{Y} \to \mathcal{X}$ of π. A N-punctured supercurve is supercurve endowed with a N-puncture. Morphisms of N-punctured supercurves over $\mathcal{Y}$ are morphisms of supercurves over $\mathcal{Y}$ commuting with the ordered sections.

Definition 2.4. A $\widetilde{N}$-puncture on a supercurve $\pi\colon \mathcal{X} \to \mathcal{Y}$ is an ordered family $(\tilde{s}_1, \ldots, \tilde{s}_N)$ of sections $\tilde{s}_i\colon Y \to X$ of the underlying projection $X \to Y$. A $\widetilde{N}$-punctured supercurve is a supercurve endowed with a $\widetilde{N}$-puncture. Morphisms of $\widetilde{N}$-punctured supercurves over $\mathcal{Y}$ are morphisms of supercurves over $\mathcal{Y}$ whose underlying ordinary morphism commutes with the ordered sections.

Note that for a supercurve over an ordinary scheme Y, N-punctures and $\widetilde{N}$-punctures coincide.

Remark 2.5. Families of N unordered points of a supercurve can be parametrized by a superscheme, the N-symmetric product (see [8,9]). They can be identified with positive superdivisors [8] or supervortices [9] of the corresponding conjugate curve.

If $\mathcal{X}$ is a superscheme, we denote by $\mathcal{X}^\bullet$ the functor of points of $\mathcal{X}$, defined by

$$\mathcal{X}^\bullet(\mathcal{Y}) = \operatorname{Hom}(\mathcal{Y}, \mathcal{X})$$

for every superscheme $\mathcal{Y}$. If $\mathcal{X}$, $\mathcal{Y}$ are superschemes, there is a one-to-one Yoneda correspondence

$$\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet),$$

where the Hom in the first member stands for morphisms of superschemes and in the second member for morphisms of functors.

Definition 2.6. A functor $\mathcal{Y} \rightsquigarrow \mathcal{F}(\mathcal{Y})$ over the category of superschemes is representable if there is a superscheme $\mathcal{M}$ and a functor isomorphism $\mathcal{M}^\bullet \xrightarrow{\sim} \mathcal{F}$.

At this point, we fix a Grothendieck topology on the category of superschemes [15]. This topology is the natural generalization of the étale topology of schemes. To this end we need the following definition.

Definition 2.7. A morphism $\mathcal{X} \to \mathcal{Y}$ of superschemes is étale if the underlying morphism of schemes $X \to Y$ is étale. A morphism $\mathcal{V} \to \mathcal{Y}$ of superschemes is an étale covering if the underlying morphism $V \to Y$ of schemes is an étale covering. The étale topology of superschemes is the Grothendieck topology whose coverings are the étale coverings of superschemes. (See, for instance, [16] for the relevant definitions.)

Definition 2.8. An étale equivalence relation of superschemes is a categorical equivalence relation $\mathcal{R} \rightrightarrows \mathcal{U}$ of superschemes whose morphisms are étale coverings [16, Definition I.5.1].

Representable functors $\mathcal{F}$ are sheaves for the étale topology in the sense that their value on a superscheme $\mathcal{Y}$ can be recovered from the values over an étale covering $\mathcal{V}$ of the superscheme and on the fiber product $\mathcal{V} \times_{\mathcal{Y}} \mathcal{V}$. This means that there is an exact sequence of sets:

$$\mathcal{F}(\mathcal{Y}) \to \mathcal{F}(\mathcal{V}) \rightrightarrows \mathcal{F}(\mathcal{V} \times_{\mathcal{Y}} \mathcal{V}),$$

that is, $\mathcal{F}(\mathcal{Y})$ is mapped isomorphically onto the set of coincidences of the two arrows $\mathcal{F}(\mathcal{V}) \rightrightarrows \mathcal{F}(\mathcal{V} \times_{\mathcal{Y}} \mathcal{V})$.

If one attempts naively to define the *functor of SUSY curves* by associating to any superscheme $\mathcal{Y}$ the set of automorphisms classes of SUSY curves $(\mathcal{X}/\mathcal{Y}, \mathcal{D})$ of genus g over it, this functor cannot be representable since it is not a sheaf. This is usually caused by the presence of automorphisms; when there are automorphisms, curves defined over different open subsets can be identified in many ways on the intersection, and then, the cocycle condition (the sheaf condition) that one needs in order to build a curve defined on the whole base scheme may fail to be fulfilled.

In the following sections we deal with this problem.

3. SUSY curves and spin curves over ordinary schemes

Let us consider, as a first step, what a right definition for the functor of SUSY curves over ordinary algebraic schemes could be. There is a morphism of functors:

$$\left\{\begin{array}{c}\text{Automorphism classes}\\ \text{of SUSY curves}\\ \text{of genus } g \text{ over } Y\end{array}\right\} \to \left\{\begin{array}{c}\text{Automorphism classes}\\ \text{of curves}\\ \text{of genus } g \text{ over } Y\end{array}\right\}$$

$$(X, \textstyle\bigwedge_{\mathcal{O}} L)/Y \mapsto X/Y$$

whose fiber functor is the functor of spin structures on X/Y, that is, the functor over the category of Y-schemes that associates to every scheme T/Y the automorphism classes of SUSY curves $(X \times_Y T, \bigwedge L)/T$.

None of these functors is a sheaf. In the case of the 'base functor', that is, the moduli functor of curves of genus g, this is due to the presence of automorphisms. This problem is circumvected by *rigidification*, that is, by fixing additional data in the curves that automorphisms must preserve. We can fix these data so that curves endowed with them have not automorphisms other than the identity. A usual choice is:

Definition 3.1. A level-n structure on a ordinary curve $\pi : X \to Y$ is an isomorphism between the n-torsion of the Jacobian variety of the curve and the group $\Gamma(Y, R^1\pi_*\mathbb{Z}_n)$.

Then a Serre's lemma [20] claims that for $n \geq 3$, an automorphism that preserves a fixed level-n structure over X/Y is the identity. From this result, one proves that for $g \geq 2$ and $n \geq 3$, the moduli functor

$$Y \rightsquigarrow \left\{ \begin{array}{c} \text{Automorphism classes of curves } X/Y \\ \text{of genus } g \text{ with a fixed level-}n\text{ structure} \end{array} \right\}$$

is representable in the category of schemes.

In the case of the 'fibre functor', that is, the functor of spin structures $T \rightsquigarrow (X \times_Y T, \bigwedge L)/T$, the problem is of different nature. We obtain a sheaf proceeding along the same line that leads to the definition of the relative Picard functor. Let us recall this definition:

Given X/Y, we can consider the functor over the category of Y-schemes that associates to every scheme $f: T \to Y$ over Y the quotient group $\operatorname{Pic}(X \times_Y T)/f^* \operatorname{Pic} T$ of the equivalence classes of invertible sheaves over $X \times_Y T$ modulo pull back of invertible sheaves over T. It is a presheaf for the étale topology, and the relative Picard functor is defined as the associated sheaf $\mathcal{P}ic_{X/Y}$ for the étale topology (see [15]). The *relative Picard group* is then defined as the group of sections of the Picard functor over Y, $\operatorname{Pic} X/Y = \mathcal{P}ic_{X/Y}(Y)$. Classes $[L]$ of invertible sheaves on X modulo $f^* \operatorname{Pic} T$ define elements in $\operatorname{Pic} X/Y$, and when there is a section $Y \hookrightarrow X$ of X/Y, one has $\gamma = [L]$ for every $\gamma \in \operatorname{Pic} X/Y$. In general an element γ may not be defined by a invertible sheaf, but there exists an étale covering $p: Y' \to Y$ such that $p^*\gamma = [L]$ for certain invertible sheaf L on $X \times_Y Y'$. Although elements in $\operatorname{Pic} X/Y$ may not be classes of invertible sheaves we shall make free use of the expression 'an invertible sheaves class in $\operatorname{Pic} X/Y$'.

The Picard functors $\mathcal{P}ic^d_{X/Y}$ of 'invertible sheaves classes of relative degree d' are representable, in the sense that there exist projective Jacobian schemes $J^d(X/Y) \to Y$ and a universal 'invertible sheaves class' $\Upsilon_d \in \operatorname{Pic}^d(X \times_Y J^d(X/Y)/J^d(X/Y))$ such that for every Y-scheme $T \to Y$ there is a functorial isomorphism

$$\begin{aligned} \operatorname{Hom}(T, J^d(X/Y)) &\rightsquigarrow \operatorname{Pic}(X \times_Y T/T) \\ \phi &\rightsquigarrow \phi^*\Upsilon_d \end{aligned}$$

(see [15]).

As a first approach to our problem, we can then consider the following alternative definition:

Definition 3.2. A spin curve over an ordinary scheme Y is a curve $X \to Y$ of genus g endowed with a level-n structure, and a 'invertible sheaves class' $\Upsilon \in \operatorname{Pic} X/Y$ such that $\Upsilon^2 = [\kappa_{X/Y}]$.

Note that when there is a section of $\pi : X \to Y$, then $\operatorname{Pic} X \to \operatorname{Pic} X/Y$ is an epimorphism and the 'relative spin structure' has the form $\Upsilon = [L]$ for some invertible sheaf L on X. However, the sheaf L may still not be a spin structure.

On the other hand, since $\pi_* \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_Y$, if there is a spin structure $L \in \Upsilon$, any other spin structure in the same class has the form $L \otimes \pi^* L'$ for an invertible sheaf L' on Y with $L'^2 = \mathcal{O}_Y$.

Definition 3.3. The functor of spin curves is the functor on the category of ordinary schemes, given by

$$Y \rightsquigarrow \mathcal{S}_{\text{spin}}(Y) = \left\{ \begin{array}{c} \text{Isomorphism classes of spin -curves } (X/Y, \Upsilon) \\ \text{of genus } g \text{ with a level-}n \text{ structure} \end{array} \right\}.$$

Similarly, the functor of $\widetilde{N}$-punctured spin -curves is defined as

$$Y \rightsquigarrow \mathcal{S}^{\widetilde{N}}_{\text{spin}}(Y) = \left\{ \begin{array}{l} \text{Isomorphism classes of } \widetilde{N}\text{-punctured} \\ \quad \text{spin curves } (X/Y, \Upsilon) \text{ of genus } g \text{ with a level-}n \text{ structure} \end{array} \right\}.$$

Theorem 3.4. *If $g \geq 2$ and $n \geq 3$, the functor of spin -curves $\mathcal{S}_{\text{spin}}$ is representable by a scheme M_{s}. There is a universal curve $X_{\text{s}}/M_{\text{s}}$ of genus g with a level-n structure, and a universal element $\Upsilon_s \in \operatorname{Pic} X_{\text{s}}/M_{\text{s}}$ with $\Upsilon_s^2 = [\kappa_{X_{\text{s}}/M_{\text{s}}}]$, such that for every scheme Y there is a functorial isomorphism*

$$\begin{aligned} \operatorname{Hom}(Y, M_{\text{s}}) &\xrightarrow{\sim} \mathcal{S}_{\text{spin}}(Y) \\ \varphi &\mapsto (\varphi^* M_{\text{s}}, \varphi^* \Upsilon_{\text{s}}). \end{aligned}$$

The moduli scheme of spin curves M_{s} is quasi-projective of dimension $3g - 3$.

Proof. The representability of the moduli functor of ordinary curves of genus g with a level-n structure means that there is a moduli scheme M and a universal curve $X_M \to M$. Let $J^d \to M$ be the relative Jacobian of invertible sheaves of degree d on $X_M \to M$ and $\Upsilon_d \in \operatorname{Pic}^d(X_M \times_M J^d/J^d)$ the universal 'invertible sheaf class'. Let $\mu_2 : J^{g-1} \to J^{2g-2}$ be the morphism 'raising to two'. The canonical sheaf $\kappa_{X_M/M}$ defines a point of J^{2g-2} with values in M, that is a morphism $M \to J^{2g-2}$ of M-schemes. If $M_{\text{s}} = \mu_2^{-1}[\kappa_{X_M/M}] \subset J^{g-1}$ is the pre-image of that point, the natural projection $M_{\text{s}} \to M$ is an étale covering of degree 2^{2g}, so that M_{s} is a quasi-projective scheme of dimension $3g - 3$.

Now, one easily proves that M_{s} endowed with the element in $\mathcal{S}_{\text{spin}}(M_{\text{s}})$ defined by the pull back $X_{\text{s}} = X_M \times_M M_{\text{s}} \hookrightarrow X_M \times_M J^{g-1}$ of the universal curve with the 'invertible sheaves class' $\Upsilon_s = \Upsilon_{g-1|X_{\text{s}}}$ represents the functor of spin curves. □

There exists an étale covering $p : U_{\text{s}} \to M_{\text{s}}$, such that $X_{U_{\text{s}}} = X_{\text{s}} \times_{M_{\text{s}}} U_{\text{s}} \to U_{\text{s}}$ has a section. Then there is an invertible sheaf L on $X_{U_{\text{s}}}$ in the class $p^* \Upsilon_{\text{s}}$. Taking eventually a finer covering, we can assume that U_s is affine and that $L^2 \xrightarrow{\sim} \kappa_s (= \kappa_{X_{U_s}/U_s})$. We can write $L = \kappa_s^{1/2}$.

Definition 3.5. A trivializing covering for M_s is an étale covering $p: U_s \to M_s$ by an affine scheme U_s, such that there exists a section $\sigma: U_s \hookrightarrow X_{U_s}$ of $X_{U_s} \to U_s$ and an invertible sheaf $\kappa_s^{1/2}$ on $X_{U_s} = X_s \times_{M_s} U_s$ with $[\kappa_s^{1/2}] = p^*\Upsilon_s$ and $(\kappa_s^{1/2})^2 = \kappa_s$.

Remark 3.6. For every trivializing covering $p: U_s \to M_s$, the fibred product $R = U_s \times_{M_s} U_s$ defines an étale equivalence relation $(p_1, p_2): U_s \times_{M_s} U_s \rightrightarrows U_s$, whose quotient (in the category of locally ringed spaces [16]) is M_s. Then, the functor of spin curves $\mathcal{S}_{\mathrm{spin}} \simeq M_s{}^\bullet$ is the quotient of equivalence relation $R^\bullet \simeq U_s{}^\bullet \times_{M_s{}^\bullet} U_s{}^\bullet \rightrightarrows U_s{}^\bullet$ in the category of sheaves of sets.

In the case of punctured spin curves, due to the presence of sections of the curves X/Y, one is forced to make a base-change to the product $X_s^N = X_s \times_{M_s} \ldots \times_{M_s} X_s$. If we consider the diagram

$$\begin{array}{ccc} \bar{X}_s^N = X_s \times_{M_s} X_s^N & \longrightarrow & X_s \\ \downarrow & & \downarrow \\ X_s^N & \longrightarrow & M_s \end{array}$$

there are N canonical sections of $\bar{X}_s^N \to X_s^N$, the N diagonals $\delta_i: X_s^N \to X_s \times_{M_s} X_s^N$, $\delta_i(x_1, \ldots, x_N) = (x_i, x_1, \ldots, x_N)$. If we endow $\bar{X}_s^N \to X_s^N$ with the pull back $\bar{\Upsilon}_s$ of the universal 'invertible sheaves class', one has:

Theorem 3.7. *If $g \geq 2$ and $n \geq 3$, the functor of $\widetilde{N}$-punctured spin curves $\mathcal{S}_{\mathrm{spin}}^{\widetilde{N}}$ is representable by the (fibred) N-power X_s^N of the universal spin curve and the element $(\bar{X}_s^N, \bar{\Upsilon}_s) \in \mathcal{S}_{\mathrm{spin}}^{\widetilde{N}}(X_s^N)$. In other words, for every scheme Y there is a functorial isomorphism*

$$\begin{aligned} \mathrm{Hom}(Y, X_s^N) &\xrightarrow{\sim} \mathcal{S}_{\mathrm{spin}}^{\widetilde{N}}(Y) \\ \bar{\varphi} &\mapsto \bar{\varphi}^*(\bar{X}_s^N/X_s^N, \bar{\Upsilon}_s, \delta_1, \ldots, \delta_N). \end{aligned}$$

4. The functor of SUSY curves

In what sequel we fix $g \geq 2$ and $n \geq 3$ and consider only curves of genus g with a level-n structure.

We have already said that only sheaves can be representable. We then define,

Definition 4.1. The functor of SUSY curves is the sheaf $\mathcal{S}$ associated to the presheaf

$$\mathcal{Y} \rightsquigarrow S(\mathcal{Y}) = \{\text{Isomorphism classes of SUSY curves over } \mathcal{Y}\}$$

for the étale topology of superschemes (Definition 2.7). Similarly, the functors of N-punctured and $\widetilde{N}$-punctured SUSY curves are the sheaves $\mathcal{S}^N$, $\mathcal{S}^{\widetilde{N}}$ in the étale topology of superschemes associated, respectively, to the presheaves:

$$\mathcal{Y} \rightsquigarrow S^N(\mathcal{Y}) = \{\text{Isomorphism classes of } N\text{-punctured SUSY curves over } \mathcal{Y}\}$$
$$\mathcal{Y} \rightsquigarrow S^{\widetilde{N}}(\mathcal{Y}) = \left\{\text{Isomorphism classes of } \widetilde{N}\text{-punctured SUSY curves over } \mathcal{Y}\right\}$$

For an ordinary scheme Y, the elements in $\mathcal{S}(Y)$ can be described in terms of spin curves. If $(\mathcal{X}/Y, \mathcal{D})$ is a SUSY curve over Y, that is, an ordinary curve X/Y with an invertible sheaf $L = (\mathcal{D} \otimes_{\mathcal{A}} \mathcal{O})^{-1}$ and a fixed isomorphism $L \otimes L \xrightarrow{\sim} \kappa_{X/Y}$, then $(X/Y, [L])$ is a spin curve in the sense of Definition 3.2. We can easily check that this defines a morphism from the presheaf of automorphism classes of SUSY curves (restricted to the category of ordinary schemes) to the sheaf $\mathcal{S}_{\text{spin}}$ of spin curves.

Theorem 4.2. *The morphism induced between the associated sheaves is an isomorphism*

$$\mathcal{S}_{|\{\text{Schemes}\}} \xrightarrow{\sim} \mathcal{S}_{\text{spin}},$$

that is, the restriction of the functor of SUSY curves to the category of ordinary schemes is the functor of spin curves.

Proof. If $(X/Y, \Upsilon)$ is a spin curve, there is an affine étale covering $p\colon U \to Y$ such that $p^*X \to U$ has a section, and then $p^*\Upsilon = [L]$. Moreover, considering eventually a finer affine covering, we can assume that L is actually a spin structure, $L \otimes L \xrightarrow{\sim} \kappa_{p^*X/U}$. This shows that the morphism of the statement is surjective.

To see that it is injective, let us consider two SUSY curves $(X/U, L)$, $(X'/U, L')$ on an affine étale covering $p\colon U \to Y$ with fixed isomorphisms $\phi\colon L \otimes L \xrightarrow{\sim} \kappa_{X/U}$ and $\phi'\colon L' \otimes L' \xrightarrow{\sim} \kappa_{X'/U}$, that define the same spin curve. We have to prove that there is an étale covering $q\colon V \to U$ such that $(q^*X/V, q^*L)$, $(q^*X'/V, q^*L')$ are isomorphic as SUSY curves over V. Notice first that $X \xrightarrow{\sim} X'$ and $[L] = [L']$; then, there is an affine étale covering $U' \to U$ such that L' and L are isomorphic after changing base to U'. We can then assume that $L' \xrightarrow{\sim} L$ so that we have two isomorphisms $\phi\colon L \otimes L \xrightarrow{\sim} \kappa_{X/U}$ and $\phi'\colon L \otimes L \xrightarrow{\sim} \kappa_{X/U}$. Now $\phi^{-1}\phi'$ is an automorphism of $L \otimes L$, and then it consists in the multiplication by an invertible element $\lambda \in \mathcal{O}_U(U)$. If $V = \operatorname{Spec} \mathcal{O}_U(U)[t]/(t^2 - \lambda)$ and $q\colon V \to U$ is the natural morphism, then q is an étale covering, $q^*\lambda = \bar{t}^2$, and $\bar{t}$ defines an automorphism of q^*L such that $q^*\phi' = q^*\phi \circ (\bar{t} \otimes \bar{t})$. □

The corresponding result for the case of punctured curves is:

Theorem 4.3. *There are natural sheaf isomorphisms*

$$\mathcal{S}^N_{|\{\text{Schemes}\}} \xrightarrow{\sim} \mathcal{S}^{\widetilde{N}}_{\{\text{Schemes}\}} \xrightarrow{\sim} \mathcal{S}^{\widetilde{N}}_{\text{spin}}\,.$$

that is, the restriction of the functors of N-punctured SUSY curves and of $\widetilde{N}$-punctured SUSY curves to the category of ordinary schemes coincide with the functor of $\widetilde{N}$-punctured spin curves. By Theorem 3.7, all these isomorphic sheaves are representable by $(\bar{X}^N_s/X^N_s, \bar{\Upsilon}_s)$.

Corollary 4.4. *There is an isomorphism*

$$\widetilde{\mathcal{S}^N} \xrightarrow{\sim} \mathcal{S} \times_{M_s^\bullet} (X_S^N)^\bullet$$

of sheaves on superschemes.

5. Towards a supermoduli of SUSY curves

We start with the study of the geometry of a supermoduli superscheme, understood as a superscheme $\mathcal{M}$ that represents the functor of SUSY curves, assuming that it exists.

One first notice that the points of $\mathcal{M}$ and M_s with values in ordinary schemes coincide. This implies that the underlying scheme to $\mathcal{M}$ is precisely M_s, thus fixing the even structure of $\mathcal{M}$.

For the odd structure of $\mathcal{M}$, if $\mathcal{M} = (M_s, \mathcal{A}_\mathcal{M})$ and $\mathcal{N} = (\mathcal{A}_\mathcal{M})_1 + (\mathcal{A}_\mathcal{M})_1^2$, then $\mathcal{A}_\mathcal{M}/\mathcal{N} \simeq \mathcal{O}_{M_s}$ and $\mathcal{E} = \mathcal{N}/\mathcal{N}^2$ is a locally free $\mathcal{O}_{M_s}$-module, such that there is locally an isomorphism $\mathcal{A}_\mathcal{M} \xrightarrow{\sim} \bigwedge_{\mathcal{O}_{M_s}} \mathcal{E}$. This means that the (local) determination of $\mathcal{E}$ provides a local description of $\mathcal{M}$.

The sheaf $\mathcal{E}$ can be computed from the identification $\mathcal{E}^* \simeq (\operatorname{Der} \mathcal{A}_\mathcal{M})_1 \otimes_{\mathcal{A}_\mathcal{M}} \mathcal{O}_{M_s} \xrightarrow{\sim} (\operatorname{Der}(\mathcal{A}_\mathcal{M}, \mathcal{O}_{M_s}))_1$ given by the exact sequence

$$0 \to \operatorname{Der} \mathcal{O}_{M_s} \to \operatorname{Der} \mathcal{A}_\mathcal{M} \otimes_{\mathcal{A}_\mathcal{M}} \mathcal{O}_{M_s} \to \mathcal{E}^* \to 0.$$

If one writes $\mathcal{O}_{M_s}[\epsilon_0, \epsilon_1] = \mathcal{O}_{M_s} \otimes_k k[\epsilon_0, \epsilon_1]$, with ϵ_i of parity i and $\epsilon_0^2 = \epsilon_1^2 = \epsilon_0\epsilon_1 = 0$, $\operatorname{Der}(\mathcal{A}_\mathcal{M}, \mathcal{O}_{M_s})$ can be identified with the space of morphisms $\mathcal{A}_\mathcal{M} \to \mathcal{O}_{M_s}[\epsilon_0, \epsilon_1]$ that induces the identity on the even part of the first component. In this identification, a derivation $D = D_0 + D_1$ (even and odd components) goes to the morphism ϕ_D described as $\phi_D(f) = f + D_0(f)\epsilon_0 + (-1)^{|f|} D_1(f)$.

In other words, $\operatorname{Der}(\mathcal{A}_\mathcal{M}, \mathcal{O}_{M_s})$ is isomorphic with the subspace of the elements in $\operatorname{Hom}(M_s[\epsilon_0, \epsilon_1], \mathcal{M}) \simeq \mathcal{S}(M_s[\epsilon_0, \epsilon_1])$ whose restriction to $\operatorname{Hom}(M_s, \mathcal{M}) = \operatorname{Hom}(M_s, M_s) \simeq \mathcal{S}(M_s) = \mathcal{S}_{\text{spin}}(M_s)$ produce the identity morphism. That is, they are infinitesimal deformations of the universal spin curve $(X_s/M_s, \Upsilon_s)$, according with the following:

Definition 5.1. The infinitesimal deformations sheaf of an element $\phi \in \mathcal{S}(Y)$ is the sheaf on Y for the étale topology given by associating to any étale covering $V \to Y$ the set $\mathcal{D}ef_{\text{Inf}}(\phi)(V)$ of the elements $\bar{\phi}_V \in \mathcal{S}(V[\epsilon_0, \epsilon_1])$ whose restriction to V is the element ϕ_V.

When the element $\phi \in \mathcal{S}(Y)$ is the image of a SUSY curve $(X/Y, \kappa^{1/2}) \in S(Y)$ by the natural morphism $S(Y) \to \mathcal{S}(Y)$ from sections of the presheaf to sections of the associated sheaf, we can also consider its infinitesimal deformations as a SUSY curve. We then define

Definition 5.2. The presheaf of infinitesimal deformations of $(X/Y, \kappa^{1/2}) \in S(Y)$ is the presheaf on Y for the étale topology that associates to an étale covering $V \to Y$ the set $\operatorname{Def}_{\text{Inf}}(X/Y, \kappa^{1/2})(V)$ of the SUSY curves $(\bar{\mathcal{X}}/V[\epsilon_0, \epsilon_1], \bar{\mathcal{D}})$ extending $(X_V/V, \kappa_V^{1/2})$.

The infinitesimal deformations in $\mathrm{Def}_{\mathrm{Inf}}(X/Y, \kappa^{1/2})(V)$ can be identified by means of the Schlessinger–Vaintrob theory [22,23] with the first cohomology group of the sheaf of infinitesimal automorphisms of $(X_V/V, \kappa_s^{1/2})$. This group is isomorphic with

$$H^1(X_V, \kappa_V^{-1} \oplus \kappa_V^{-1/2})$$

(see [11,12,18]). It follows easily that the presheaf $\mathrm{Def}_{\mathrm{Inf}}(X/Y, \kappa^{1/2})$ is actually a sheaf. Moreover, one has:

Lemma 5.3. *There is a natural sheaf isomorphism,*

$$\mathrm{Def}_{\mathrm{Inf}}(X/Y, \kappa^{1/2}) \xrightarrow{\sim} \mathcal{D}ef_{\mathrm{Inf}}(\phi)\,,$$

where $\phi \in \mathcal{S}(Y)$ is the image of the SUSY curve $(X/Y, \kappa^{1/2}) \in \mathcal{S}(Y)$.

Now, we turn back to the determination of the infinitesimal deformations of the universal spin-curve $(X_s/M_s, \Upsilon_s)$.

Let us consider a trivializing covering $p: U_s \to M_s$ (Definition 3.5). We fix a spin structure $\kappa_s^{1/2}$ on X_{U_s} such that $[\kappa_s^{1/2}] = p^*\Upsilon_s$.

In order to compute locally the infinitesimal deformations of $(X_s/M_s, \Upsilon_s)$, we have to compute the infinitesimal deformations of the element $\phi \in \mathcal{S}(U_s)$ defined by the SUSY curve $(X_{U_s}/U_s, \kappa_s^{1/2}) \in \mathcal{S}(U_s)$. By the above considerations:

$$\mathcal{D}ef_{\mathrm{Inf}}(\phi)(U_s) = H^1(X_{U_s}, \kappa_s^{-1} \oplus \kappa_s^{-1/2}) \xrightarrow{\sim} \Gamma(U_s, R^1\pi_*(\kappa_s^{-1} \oplus \kappa_s^{-1/2}))\,,$$

where $\pi: X_{U_s} \to U_s$ is the projection. The last isomorphism holds because U_s is affine.

Since the sections of $\mathcal{E}^*$ are the odd part of the space $\mathcal{D}ef_{\mathrm{Inf}}(\phi)(U_s)$ of these deformations, $\mathcal{E}$ can be locally identified with $\left(R^1\pi_*(\kappa_s^{-1/2})\right)^* \xrightarrow{\sim} \pi_*(\kappa_s^{3/2})$.

Definition 5.4. The superscheme of local spin–gravitino fields on a trivializing covering $p: U_s \to M_s$ is the superscheme

$$\mathcal{U}_s = (U_s, \bigwedge \pi_*\kappa_s^{3/2})\,,$$

where $\kappa_s^{3/2} = \kappa_s^{1/2} \otimes \kappa_s$ for a spin structure $\kappa_s^{1/2}$ on X_{U_s} such that $[\kappa_s^{1/2}] = p^*\Upsilon_s$. This superscheme has dimension $(3g-3, 2g-2)$.

The superscheme of local spin–gravitino fields on a trivializing covering of M_s is then the natural candidate for a 'local supermoduli' of SUSY curves. Using a generalization of Kodaira–Spencer theory [17] for deformations of SUSY curves, LeBrun and Rothstein proved [18] that this is actually true. We quote some results in [18] for further use:

Theorem 5.5. *Let $\pi: X \to V$ be an ordinary curve over an affine scheme whose classical Kodaira–Spencer map $ks(\pi)$ is an isomorphism. Then for every SUSY curve $(X, \mathcal{O} \oplus \kappa^{1/2})$ over V, there is a SUSY curve $(\mathcal{X}, \mathcal{D})$ over the superscheme $(V, \bigwedge \pi_*\kappa^{3/2})$ extending $(X, \mathcal{O} \oplus \kappa^{1/2})$ and whose Kodaira–Spencer map $ks(\bar{\pi})$ is an isomorphism.*

Theorem 5.6. *Let $(\mathcal{X}/\mathcal{V}, \mathcal{D})$ be a SUSY curve whose Kodaira–Spencer map $ks(\bar{\pi})$ restricts to an isomorphism on the underlying SUSY curve $(X, \mathcal{O} \oplus \kappa^{1/2})$ over the ordinary scheme V. For every morphism of schemes $\varphi\colon Y \to V$ there is a one-to-one correspondence*

$$\left\{\begin{matrix} \textit{morphisms of superschemes} \\ \mathcal{Y} \to \mathcal{V} \textit{ extending} \\ \varphi\colon Y \to V \end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix} \textit{classes of SUSY curves} \\ (\mathcal{X}/\mathcal{Y}, \mathcal{D}) \textit{ extending} \\ \varphi^*(X, \mathcal{O} \oplus \kappa^{1/2})/Y \end{matrix}\right\}$$
$$\phi \mapsto (\phi^*\mathcal{X}/\mathcal{Y}, \phi^*\mathcal{D}).$$

6. The representability theorem for SUSY curves

In this section we give a global description of the functor $\mathcal{S}$ of SUSY curves in terms of the superscheme $\mathcal{U}_s$ of local spin-gravitino fields on a trivializing covering $p\colon U_s \to M_s$ (Definition 5.4) associated to a fixed spin structure $k_s^{1/2}$ in $p^*\Upsilon_s$. As we said before, we are fixing $g \geq 2$ and $n \geq 3$ and considering only curves of genus g with a level-n structure.

Lemma 6.1. *For every trivializing covering $p\colon U_s \to M_s$, there is an isomorphism of sheaves*

$$\mathcal{S} \times_{M_s^\bullet} U_s^\bullet \simeq \mathcal{U}_s^\bullet .$$

Proof. Let S be the presheaf functor of SUSY curves (Definition 4.1) and $S_{|\{\text{Schemes}\}}$ its restriction to the category of ordinary schemes. There is a natural presheaf morphism $U_s^\bullet \to S_{|\{\text{Schemes}\}}$, that maps a scheme morphism $\varphi\colon Y \to U_s$ to the SUSY curve over Y obtained by pull back of the SUSY curve $(X_{U_s}/U_s, k_s^{1/2})$.

We can then consider the fibred product $S \times_{S_{|\{\text{Schemes}\}}} U_s^\bullet$ as a presheaf over the category of superschemes. A section of this presheaf on a superscheme $\mathcal{Y}$ is merely the pair given by a SUSY curve $(\mathcal{X}_\mathcal{Y}/\mathcal{Y}, \mathcal{D}_\mathcal{Y})$ and a morphism of schemes $\varphi\colon Y \to U_s$, such that $(\mathcal{X}_\mathcal{Y}/\mathcal{Y}, \mathcal{D}_\mathcal{Y})$ is an extension of $\varphi^*(X_{U_s}, \mathcal{O} \oplus k_s^{1/2})$.

The first member in the statement is the sheaf associated to this presheaf. Let us now describe the second member. Since the Kodaira–Spencer map of the ordinary curve (X_{U_s}/U_s) is an isomorphism, we can apply Theorem 5.5 to the SUSY curve $(X_{U_s}, \mathcal{O} \oplus k_s^{1/2})$ over U_s, to obtain a SUSY curve

$$(\mathcal{X}_{\mathcal{U}_s}/\mathcal{U}_s, \mathcal{D}_s)$$

on the superscheme $\mathcal{U}_s = (U_s, \bigwedge \pi_*\kappa_s^{3/2})$ extending $(X_{U_s}, \mathcal{O} \oplus \kappa_s^{1/2})$ and whose Kodaira–Spencer map is an isomorphism. Then, by Theorem 5.6, there is a presheaf isomorphism

$$\mathcal{U}_s^\bullet \xrightarrow{\sim} S \times_{S_{|\{\text{Schemes}\}}} U_s^\bullet .$$

Since $U_s^\bullet$ is a sheaf, the second member is also a sheaf, and it then coincides with its associated sheaf $\mathcal{S} \times_{M_s^\bullet} U_s^\bullet$. □

We can then identify $\mathcal{S} \times_{M_s^\bullet} U_s^\bullet \times_{M_s^\bullet} U_s^\bullet \simeq \mathcal{U}_s^\bullet \times_{M_s^\bullet} U_s^\bullet$ and define an equivalence relation

$$\mathcal{U}_s^\bullet \times_{M_s^\bullet} U_s^\bullet \rightrightarrows \mathcal{U}_s^\bullet$$

as the trivial extension of the equivalence relation $(p_1, p_2)\colon U_s \times_{M_s} U_s \rightrightarrows U_s$. Since there is a categorical quotient $U_s \times_{M_s} U_s \rightrightarrows U_s \to M_s$, we obtain:

Theorem 6.2. *The functor $\mathcal{S}$ of SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure is the categorical quotient*

$$\mathcal{R}^\bullet = (\mathcal{U}_s \times_{M_s} U_s)^\bullet \rightrightarrows \mathcal{U}_s{}^\bullet \to \mathcal{S}$$

in the category of sheaves of sets on superschemes.

The functor of SUSY curves $\mathcal{S}$ can be then understood as a quotient of an étale equivalence relation of superschemes (Definition 2.8). But even in the category of schemes, étale equivalence relations may fail to have a categorical quotient [16]. This problem is solved with the introduction of Artin's algebraic spaces [1,16], which are natural outgrows of schemes. In this larger category any étale equivalence relation has a categorical quotient [16]. In particular, the quotient of an étale equivalence relation of schemes always exists as an algebraic space. We then mimic the definition of Artin's algebraic space as follows:

Definition 6.3. An Artin's algebraic superspace is a sheaf $\mathcal{F}$ on the category of superschemes for the étale topology such that there is an étale equivalence relation $\mathcal{R} \rightrightarrows \mathcal{U}$ of superschemes (Definition 2.8) and a sheaf morphism $\mathcal{U}^\bullet \to \mathcal{F}$ that induces a sheaf isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{F}$ where $\mathcal{G}$ is the quotient sheaf of the induced equivalence relation $\mathcal{R}^\bullet \rightrightarrows \mathcal{U}^\bullet$. A morphism of Artin's algebraic superspaces is merely a natural transformation of functors.

We have that $\mathcal{U}^\bullet \times_{\mathcal{F}} \mathcal{U}^\bullet \xrightarrow{\sim} \mathcal{R}^\bullet$, and we shall say that $\mathcal{U}^\bullet \to \mathcal{F}$ is a *representable étale covering* of Artin's algebraic superspace $\mathcal{F}$. The *dimension* of an algebraic superspace $\mathcal{F}$ is the dimension of $\mathcal{U}$ for a representable étale covering $\mathcal{U}^\bullet$ (it is independent of the choice of $\mathcal{U}$).

In particular, the functor of the points $\mathcal{X}^\bullet$ of a superscheme $\mathcal{X}$ is an Artin's algebraic superspace, and $\mathcal{X} \rightsquigarrow \mathcal{X}^\bullet$ embeds the category of superschemes as a full subcategory of the category of Artin's algebraic superspaces, that is, $\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) = \operatorname{Hom}(\mathcal{X}^\bullet, \mathcal{Y}^\bullet)$.

Let $\mathcal{F}$ be an Artin's algebraic superspace and $\mathcal{U}^\bullet \to \mathcal{F}$ a representable étale covering. The étale equivalence relation $\mathcal{R} \rightrightarrows \mathcal{U}$ induces an étale equivalence relation $R \rightrightarrows U$ between the underlying ordinary schemes.

Definition 6.4. The underlying Artin's algebraic space F to $\mathcal{F}$ is the quotient sheaf of $R^\bullet \rightrightarrows U^\bullet$.

In what sequel, we shall identify a superscheme $\mathcal{X}$ with its corresponding Artin's algebraic superspace $\mathcal{X}^\bullet$ and write simply $\mathcal{X}$ for any of them.

Theorem 6.2 can be now restated as a representability theorem:

Theorem 6.5. *The functor $\mathcal{S}$ of SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure is representable by an Artin's algebraic superspace whose underlying Artin's algebraic space is the moduli scheme M_s of spin curves. That is, there is a fine moduli superspace for SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure, and it is an Artin's algebraic superspace of dimension* $(3g-3, 2g-2)$.

Let us consider the exact sequence of algebraic spaces

$$\mathcal{R} \rightrightarrows \mathcal{U}_s \to \mathcal{S},$$

given by Theorem 6.2. If $(\mathcal{X}_{\mathcal{U}_s}/\mathcal{U}_s, \mathcal{D}_s)$ is the SUSY curve obtained in the proof of Lemma 6.1, there is a commutative diagram of algebraic superspaces

$$\begin{array}{ccccc} \mathcal{X}_{\mathcal{R}} & \rightrightarrows & \mathcal{X}_{\mathcal{U}_s} & \longrightarrow & \mathcal{X}_{\mathcal{S}} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{R} & \rightrightarrows & \mathcal{U}_s & \longrightarrow & \mathcal{S} \end{array}$$

where $\mathcal{X}_{\mathcal{R}} = \mathcal{X}_{U_s} \times_{M_s} U_s$ and $\mathcal{X}_{\mathcal{S}}$ is the algebraic superspace obtained as the quotient of the étale equivalence relation $\mathcal{X}_{\mathcal{R}} \rightrightarrows \mathcal{X}_{\mathcal{U}_s}$ of superschemes.

We easily prove that:

Theorem 6.6. *The sheaf functor $\mathcal{S}^N$ of N-punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure is representable by Artin's algebraic superspace $\mathcal{X}_{\mathcal{S}}^N$. That is, there is a fine moduli superspace for N-punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure and it is an Artin's algebraic superspace of dimension* $(3g-3+N, 2g-2+N)$.

If we wish to consider only punctures corresponding to different points, we would have to restrict to the open subset of $\mathcal{X}_{\mathcal{S}}^N$ complementary of all the diagonals.

In the case of $\widetilde{N}$-punctured SUSY curves, we prove directly from Theorem 6.2 that:

Theorem 6.7. *The sheaf functor $\mathcal{S}^{\widetilde{N}}$ of $\widetilde{N}$-punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure is representable by Artin's algebraic superspace $\mathcal{S} \times_{M_s} X_s^N$. That is, there is a fine moduli superspace for $\widetilde{N}$-punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure and it is an Artin's algebraic superspace of dimension* $(3g-3+N, 2g-2)$.

The underlying Artin's algebraic space to $\mathcal{S} \times_{M_s} X_s^N$ is the moduli scheme X_s^N of $\widetilde{N}$-punctured spin curves of genus $g \geq 2$ with a level-$n \geq 3$ structure.

7. The representability theorem for $\widetilde{N}$-punctured SUSY curves

An interesting question that arises now is to elucidate whether the algebraic superspaces $\mathcal{S}$ and $\mathcal{S} \times_{M_s} X_s^N$ are superschemes. This question is still pending in the case of the moduli algebraic superspace $\mathcal{S}$ of SUSY curves, but as we shall see in this section, there is an affirmative answer in the case of $\widetilde{N}$-punctured SUSY curves. Moreover, it turns out that the superscheme representing the functor of $\widetilde{N}$-punctured SUSY curves is actually *split*.

Let us start by recalling that due to Theorem 4.3, there is a universal $\widetilde{N}$-punctured spin -curve $(\bar{X}_s^N/X_s^N, \bar{\Upsilon}_s)$ on the scheme X_s^N. Now, let us consider the relative Jacobian $J^{g-1} = J^{g-1}(X_s/M_s)$ and its canonical theta-divisor $\Theta = W_{g-1}$.

Lemma 7.1. *There exists an invertible sheaf $\bar{L}$ on $\bar{X}_s^N$ in the class of the universal element $\bar{\Upsilon}_s$ that can be constructed from Θ in a natural way.*

Proof. One can assume $N = 1$, since there is a projection $X_s^N \to X_s$. Let $\bar{J}^d = J^d(\bar{X}_s/X_s)$. There is a commutative diagram

$$\begin{array}{ccccc} \bar{X}_s & \xrightarrow{\tau} & \bar{J}^{g-1} & \longrightarrow & J^{g-1} \\ \downarrow & & \downarrow & & \downarrow \\ X_s & = & X_s & \longrightarrow & M_s \end{array}$$

where $\tau: \bar{X}_s \hookrightarrow \bar{J}^{g-1}$ is the closed immersion defined by the diagonal $\delta: X_s \to X_s \times_{M_s} X_s$; for closed points, $\tau(x, x_1)$ is the divisor $(x + (g-2)x_1, x_1)$ in the fiber of x_1.

If $\Psi: \bar{X}_s^N \to J$ is the composition morphism given by the top row of the above diagram, one easily checks that for every point $x_1 \in X_s$ the restriction of $\Psi^*(\Theta)$ to the fibre of x_1 has the form $\Psi^*(\Theta)_{|X_s \times \{x_1\}} = \bar{D} + x_1$, where $\bar{D}$ is a semi-canonical divisor, $2\bar{D} = K$. The invertible sheaf $\bar{L} = \mathcal{O}(\Psi^*(\Theta)) \otimes \mathcal{O}(\delta(X_s))^{-1}$ fulfills the requirements. □

The invertible sheaf $\bar{L}$ may fail to be a spin structure, but it is a canonical representant in the universal class $\bar{\Upsilon}_s$. We can then consider the following superscheme that remains the superscheme of local spin–gravitino fields (Definition 5.4).

Definition 7.2. The superscheme of $\widetilde{N}$-punctured spin–gravitino fields is the split superscheme

$$\mathcal{X}_s^N = (X_s^N, \textstyle\bigwedge \pi_*(\bar{L} \otimes \kappa)),$$

where $\kappa = \kappa_{\bar{X}_s^N/X_s^N}$ and π is the projection $\bar{X}_s^N \to X_s^N$. It has dimension $(3g-3+N, 2g-2)$.

Our next aim is to prove that the superscheme $\mathcal{X}_s^N$ is a moduli superscheme for the functor $\widetilde{\mathcal{S}^N} \equiv \mathcal{S} \times_{M_s} X_s^N$ of $\widetilde{N}$-punctured SUSY curves. For the sake of simplicity, we first consider the case $N = 1$.

Lemma 7.3. *Given a trivializing covering $p: U_s \to M_s$ of the moduli scheme M_s of spin curves, there is a étale covering $V_s \to X_{U_s} = p^*(X_s)$ such that there is an isomorphism*

$$\widetilde{\mathcal{S}^1} \times_{X_s} V_s \equiv (\mathcal{S} \times_{M_s} X_s) \times_{X_s} V_s \simeq \mathcal{X}_s \times_{X_s} V_s$$

of Artin's algebraic superspaces, commuting with the natural projections to V_s.

Proof. Let us write $\bar{X}_{U_s} \equiv \bar{X}_s \times_{M_s} \times U_s = X_s \times_{M_s} \times X_{U_s}$. The pull backs to $\bar{X}_{U_s}/X_{U_s}$ of $(\bar{X}_s/X_s, \bar{L})$ and $(X_{U_s}/U_s, \kappa_s^{1/2})$ are in the pull back of the universal element $\bar{\Upsilon}_s$. Since both define the same section of the sheaf of spin curves, there is an affine étale covering $V_s \to X_{U_s}$ such that both become isomorphic spin curves after such a base-change. Let us denote by (X_{V_s}, L_s) any of these isomorphic SUSY curves, which can be considered as a punctured SUSY curve by means of the canonical section δ_s induced by the diagonal of $(\bar{X}_s/X_s)$.

Since the Kodaira–Spencer map of the curve (X_{V_s}/V_s) is an isomorphism, we can apply Theorem 5.5 to obtain a SUSY curve

$$(\mathcal{X}_{\mathcal{V}_s}/\mathcal{V}_s, \mathcal{D}_s)$$

over the superscheme $\mathcal{V}_s = (V_s, \bigwedge \pi_*(L_s \otimes \kappa_{X_{V_s}/V_s}))$ extending $(X_{V_s}, \mathcal{O} \oplus L_s)$ and that fulfills the hypotheses of Theorem 5.6.

Proceeding as in the proof of Lemma 6.1, we define an isomorphism

$$\Psi : \mathcal{S} \times_{M_s} V_s \xrightarrow{\sim} \mathcal{V}_s$$

of sheaves on superschemes (actually of algebraic superspaces) that commutes with the projections onto V_s. The natural identifications $\mathcal{S} \times_{M_s} V_s \simeq (\mathcal{S} \times_{M_s} X_s) \times_{X_s} V_s$ and $\mathcal{V}_s \simeq \mathcal{X}_s \times_{X_s} V_s$ enable us to conclude the proof. □

As a consequence, the equivalence relation $V_s \times_{X_s} V_s \rightrightarrows V_s \to X_s$ induces a commutative diagram

$$\begin{array}{ccccc} (\mathcal{S} \times_{M_s} X_s) \times_{X_s} V_s \times_{X_s} V_s & \rightrightarrows & (\mathcal{S} \times_{M_s} X_s) \times_{X_s} V_s & \to & \mathcal{S} \times_{M_s} X_s \\ \wr\downarrow & & \wr\downarrow & & \\ \mathcal{X}_s \times_{X_s} V_s \times_{X_s} V_s & \rightrightarrows & \mathcal{X}_s \times_{X_s} V_s & \to & \mathcal{X}_s. \end{array}$$

We have then:

Theorem 7.4. *The functor $\mathcal{S} \times_{M_s} X_s$ of punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure, is representable by the $(3g-2, 2g-2)$-dimensional superscheme $\mathcal{X}_s$ of punctured spin–gravitino fields.*

The case $N > 1$ is completely analogous and it leads to:

Theorem 7.5. *The functor $\mathcal{S}^{\widetilde{N}}$ of $\widetilde{N}$-punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure is representable by the $(3g-3+N, 2g-2)$-dimensional superscheme $\mathcal{X}_s^N$ of $\widetilde{N}$-punctured spin–gravitino fields.*

This theorem says that there is a moduli superscheme $\mathcal{X}_s^N$ for $\widetilde{N}$-punctured SUSY curves of genus $g \geq 2$ with a level-$n \geq 3$ structure. Since the moduli superscheme $\mathcal{X}_s^N$ is actually split, this gives a partial answer to the splitness question for the moduli of SUSY curves.

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